Gauge-invariant fluctuations of the metric in stochastic inflation

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I derive the stochastic equation for the perturbations of the metric for a gauge-invariant energy-momentum tensor in stochastic inflation. A quantization for the field that describes the gauge-invariant perturbations for the metric is developed. In a power-law expansion for the universe the amplitude for these perturbations on a background metric could be very important in the infrared sector.

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During inflation vacuum fluctuations on scales less than the Hubble radius are magnified into classical perturbations in the scalar fields on scales larger than the Hubble radius. These classical perturbations in the scalar fields can then change the number of e folds of expansion and so lead to classical curvature and density perturbations after inflation. These density perturbations are thought to be responsible for the formation of galaxies and the large scale structure of the observable universe as well as, in combination with the gravitational waves produced during inflation, for the anisotropies in the cosmic microwave background.

In this report I consider the gauge-invariant fluctuations of the metric on a globally flat Friedmann-Robertson-Walker (FRW) metric in the early inflationary universe. These metric fluctuations are here considered in the framework of the linear perturbative corrections. A nonlinear perturbative calculation for this issue was developed in [1,2]. The scalar metric perturbations of the metric are associated with density perturbations. These are the spin-zero projections of the graviton, which only exist in nonvacuum cosmologies. The issue of gauge invariance becomes critical when we attempt to analyze how the scalar metric perturbations produced in the very early universe influence of a globally flat isotropic and homogeneous universe. This allows us to formulate the problem of the amplitude for the scalar metric perturbations on the evolution of the background FRW universe in a coordinate-independent manner at every moment in time. Since the results do not depend on the gauge, the perturbed globally flat isotropic and homogeneous universe is well described by [3]

$$ds^2 = (1+2\psi)dt^2 - a^2(t)(1-2\chi)dx^2, \qquad (1)$$

where a is the scale factor of the universe and ψ and χ the perturbations of the metric. I will consider the particular case where the tensor T_{ij} is diagonal, i.e., for $\chi = \psi$ [5]. As in a previous work [4] I consider a semiclassical expansion for the scalar field $\varphi(\vec{x},t) = \phi_c(t) + \phi(\vec{x},t)$, with expectation values $\langle E|\varphi|E\rangle = \phi_c(t)$ and $\langle E|\phi|E\rangle = 0$. Here, $|E\rangle$ is a unknown state of the universe. Because of $\langle E|\chi|E\rangle = 0$, the expectation value of the metric (1) gives the background metric that describes a flat FRW spacetime. Linearizing the

Einstein equations in terms of ϕ and χ, one obtains the system of differential equations for ϕ and χ:

$$\ddot{\chi} + \left(\frac{\dot{a}}{a} - 2\frac{\ddot{\phi}_c}{\dot{\phi}_c}\right)\dot{\chi} - \frac{1}{a^2}\nabla^2\chi + \left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{\dot{a}}{a}\frac{\ddot{\phi}_c}{\dot{\phi}_c}\right]\chi = 0,$$

$$(2)$$

$$\frac{1}{a}\frac{d}{dt}(a\chi)_{,\beta} = \frac{4\pi}{M_p^2}(\dot{\phi}_c\phi)_{,\beta}, \qquad (3)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V''(\phi_c)\phi + 2V'(\phi_c)\chi - 4\dot{\phi}_c\dot{\chi} = 0.$$

$$(4)$$

Here, the dynamics of ϕ_c being described by the equations

$$\ddot{\phi}_c + 3\frac{\dot{a}}{a}\dot{\phi}_c + V'(\phi_c) = 0, \qquad (5)$$

$$\dot{\phi}_c = -\frac{M_p^2}{4\pi}H_c'(\phi_c), \qquad (6)$$

where the prime denotes the derivative with respect to ϕ_c and $H_c(\phi_c) \equiv \dot{a}/a$. Equation (2) for χ can be simplified with the map $h = e^{1/2\int(\dot{a}/a - 2\ddot{\phi}_c/\dot{\phi}_c)dt}\chi$

$$\ddot{h} - \frac{1}{a^2}\nabla^2 h + \left\{\frac{1}{4}\left(\frac{\dot{a}}{a} - 2\frac{\ddot{\phi}_c}{\dot{\phi}_c}\right)^2 \right.$$

$$-\frac{1}{2}\left(\frac{\ddot{a}a - \dot{a}^2}{a^2} - \frac{2\frac{d}{dt}(\ddot{\phi}_c\dot{\phi}_c) - 4\dot{\phi}_c^2}{\dot{\phi}_c^2}\right)$$

$$\left. + \left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{\dot{a}}{a}\frac{\ddot{\phi}_c}{\dot{\phi}_c}\right]\right\}h = 0. \qquad (7)$$

The Eq. (7) is a Klein-Gordon equation for the redefined fluctuations of the metric $h(\vec{x},t)$ in a curved spacetime de-

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fined with a flat FRW metric for the background. This field can be written as a Fourier expansion in terms of the modes $h_{\vec{k}}=e^{i\vec{k}\cdot\vec{x}}\xi_k(t)$

$$h(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k[a_k h_k + a_k^\dagger h_k^*], \qquad (8)$$

where a_k and $a_k^\dagger$ are the annihilation and creation operators with commutation relations $[a_{\vec{k}},a_{\vec{k}'}^\dagger]=\delta^{(3)}(\vec{k}-\vec{k}')$, and the asterisk denotes the complex conjugate. The matter field perturbations, written as a Fourier expansion, is

$$\phi(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k[a_k \phi_k + a_k^\dagger \phi_k^*], \qquad (9)$$

where $\phi_{\vec{k}}=e^{i\vec{k}\cdot\vec{x}}u_k(t)$. Due to the fact $\chi=\psi$, the metric and matter perturbations are anticorrelated outside the horizon $\xi_k=-\phi_c(t)\,e^{-1/2\int(\dot{a}/a-2\ddot{\phi}_c/\dot{\phi}_c)dt}\,u_k$ [5]. The equation for the time-dependent modes $\xi_k(t)$ is

$$\ddot{\xi}_k+\omega_k^2(t)\xi_k=0, \qquad (10)$$

where $\omega_k(t)$ is the time-dependent frequency for each mode with wave number k: $\omega_k^2(t)=[k^2/a^2-k_o^2/a^2]$. Here, $k_o(t)$ is the time dependent wave number that separates the infrared (IR) and the ultraviolet (UV) sectors. On super Hubble scales, $k^2/a^2\ll k_o^2/a^2$ and Eq. (10) for the time dependent frequencies become k independent: $\omega_k^2(t)\simeq -k_o^2(t)/a^2$. The commutation relation for h and $\dot{h}$ is $[h(\vec{x},t),\dot{h}(\vec{x}',t)]=i\delta^{(3)}(\vec{x}-\vec{x}')$ for $\xi_k\dot{\xi}_k^*-\dot{\xi}_k\xi_k^*=i$. When the modes become real one obtains $\xi_k\dot{\xi}_k^*-\dot{\xi}_k\xi_k^*=0$ and the field h is classical [6].

Stochastic approach. Now I consider the field $h(\vec{x},t)$ on the IR sector. For $k\ll k_o(t)$ the coarse-grained fields $h_{CG}(\vec{x},t)$ and $\phi_{CG}(\vec{x},t)$ can be written as

$$h_{CG}(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\,\theta(\epsilon k_o-k)[a_k h_k + a_k^\dagger h_k^*], \qquad (11)$$

$$\phi_{CG}(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\,\theta(\epsilon k_o-k)[a_k \phi_k + a_k^\dagger \phi_k^*], \qquad (12)$$

where $\epsilon\ll 1$ is an dimensionless constant. Replacing Eq. (11) in Eq. (7), one obtains the following stochastic equation for h_{CG}:

$$\ddot{h}_{CG}-\frac{k_o^2}{a^2}h_{CG}=\epsilon\left[\frac{d}{dt}(\dot{k}_o\eta)+2\dot{k}_o\kappa\right], \qquad (13)$$

where the noises η and κ are

$$\eta(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\,\delta(\epsilon k_o-k)[a_k h_k + a_k^\dagger h_k^*], \qquad (14)$$

$$\kappa(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\,\delta(\epsilon k_o-k)[a_k \dot{h}_k + a_k^\dagger \dot{h}_k^*]. \qquad (15)$$

The fluctuations h_{CG} will be classical if all the modes ξ_k (for $k\ll k_o$) are real [6]. Thus, when

$$\left|\frac{\mathrm{Im}(\xi_k)}{\mathrm{Re}(\xi_k)}\right|\ll 1, \qquad (16)$$

for $0<k<\epsilon k_o$, Eq. (13) is a classical stochastic equation that describes the gauge-invariant redefined fluctuations on the IR sector. When $(\dot{k}_o)^2\langle\kappa^2\rangle\ll(\ddot{k}_o)^2\langle\eta^2\rangle$, one can neglect the noise κ with respect to η in Eq. (13). In this case one obtain the following two first order stochastic equations for h_{CG}:

$$\dot{h}_{CG}=\epsilon\dot{k}_o\eta+u, \qquad (17)$$

$$\dot{u}=\frac{k_o^2}{a^2}h_{CG}, \qquad (18)$$

where u is an auxiliar field. The Fokker-Planck equation for the system (18) gives the transition probability $P(h_{CG},u,t|h_{CG}^{(0)},u^{(0)},t_o)$ for the universe, from the initial configuration $(h_{CG}^{(0)},u^{(0)},t_o)$ to the (h_{CG},u,t) one

$$\frac{\partial P}{\partial t}=-u\frac{\partial P}{\partial h_{CG}}-\frac{k_o^2}{a^2}h_{CG}\frac{\partial P}{\partial u}+\frac{\epsilon^3\dot{k}_o k_o^2}{4\pi}\xi_{\epsilon k_o}^2\left[\frac{\partial^2 P}{\partial h_{CG}^2}\right]. \qquad (19)$$

Heisenberg representation for h_{CG}. Equation (13) can be written as

$$\ddot{h}_{CG}-\left[\frac{k_o(t)}{a(t)}\right]^2 h_{CG}+\xi_c(\vec{x},t)=0, \qquad (20)$$

where $\xi_c(\vec{x},t)=-\epsilon[(d/dt)(\dot{k}_o\eta^{(c)})+2\dot{k}_o\kappa]$. This noise becomes from the short wavelength sector due to the cosmological evolution of both, the horizon and the scale factor of the universe. The effective Hamiltonian associated with Eq. (20) is

$$H_{\mathrm{eff}}(h_{CG},t)=\frac{1}{2}P_{CG}^2+\frac{1}{2}\mu^2(t)(h_{CG})^2+\xi_c h_{CG}, \qquad (21)$$

where $P_{CG}\equiv\dot{h}_{CG}$ and $\mu^2(t)=k_o^2/a^2$. Note that ξ_c plays the role of an external classical stochastic force in the effective Hamiltonian (21). Thus, one can write the following Schrödinger equation:

$$i\frac{\partial}{\partial t}\Psi(h_{CG},t)=-\frac{1}{2}\frac{\partial^2}{\partial(h_{CG})^2}\Psi(h_{CG},t)$$
$$+\left[\frac{1}{2}\mu^2(t)(h_{CG})^2+\xi_c h_{CG}\right]\Psi(h_{CG},t),$$

$$(22)$$

where $\Psi(h_{CG},t)$ is the wave function that characterize the system. Observe that generally $\mu(t)$ depends on time, and the Hamiltonian (21) is nonconservative, also in the case in which one would neglect the stochastic force. The only case where μ does not present time dependence is in a de Sitter expansion of the universe. In this case μ is constant and Eq. (21) represents a harmonic oscillator with a stochastic external force ξ_c. In this case we have a forced linear harmonic oscillator and the solution is a coherent state with the displacement due to the action of the external force. The effective Hamiltonian (21) describes an open system. This is due to the fact that degrees of freedom of the infrared sector are constantly increasing since the k_o-temporal dependence.

The probability to find the universe with a given h_{CG} in a given time t is

$$P(h_{CG},t)=\Psi(h_{CG},t)\Psi^*(h_{CG},t),\qquad(23)$$

where the asterisk denotes the complex conjugate.

Power-law expansion. Now I study the particular case of a power-law expansion for the universe. In this case the scale factor is $a(t)\propto(t/t_o)^p$, and the Hubble parameter $H_c[\phi_c(t)]=p/t$. The temporal evolution of the background field ϕ_c is $\phi_c(t)=\phi_c^{(o)}-m\ln[(t/t_o)p]$, where $\phi_c^{(o)}\equiv\phi_c(t=t_o)$ — for $t\geq t_o$. Here, $m\simeq(10^{-4}-10^{-6})\,M_p$ is the mass of the inflaton field. The map for the redefined fluctuations h become $h(\vec{x},t)=t^{(p/2+1)}\chi(\vec{x},t)$. Furthermore, one obtains

$$\frac{k_o^2}{a^2}=-\frac{1}{4}(p^2+6p+4)t^{-2}.\qquad(24)$$

In the IR sector one obtains the wavelength are greater than the size of the horizon [i.e., $k^{-1}\gg k_o^{-1}(t)$] and the term k^2/a^2 can be neglected with respect to the another one k_o^2/a^2 in the equation for the temporal modes ξ_k. Hence, the general solution for ξ_k is

$$\xi_k(t)\simeq C_1 t^{1/2(1-i\sqrt{|1-4K^2|})}+C_2 t^{1/2(1+i\sqrt{|1-4K^2|})},\quad(25)$$

where $K^2=1/4(p^2+6p+4)$. It is well known that the fluctuations in the infrared sector become classical. Thus, I will consider that the condition $\xi_k\dot{\xi}_k^*-\dot{\xi}_k\xi_k^*=0$ holds in this sector. This implies that $C_1=\pm C_2$. I consider the case where the universe expands very rapidly ($p\gg1$)

Case (1), $C_1=-C_2$: in this case the time dependent modes are (for $C_1=-i|C_1|$)

$$\xi_k^{(1)}(t)\simeq2|C_1|\sin[\omega(t)t],\qquad(26)$$

where

$$\omega(t)=\frac{\sqrt{|1-4K^2|}}{2t}\ln[t],\qquad(27)$$

is the frequency of the scalar perturbations of the metric, which depends on time. *Case (2)*, $C_1=C_2$: here the modes are (for $C_1=|C_1|$)

$$\xi_k^{(2)}(t)\simeq2|C_1|t^{1/2}[1+t^{-1/2}\cos[\omega(t)t]],\qquad(28)$$

where $\omega(t)$ is given by Eq. (27). Note that $\lim_{t\to\infty}\omega(t)\to0$ and $\lim_{t\to0}\omega(t)\to\infty$ in both cases (1) and (2). Case (1) describes an oscillatory scalar perturbations of the metric with constant amplitude $|C_1|$ and an oscillation frequency that decreases with time. This means that for very large t the scalar perturbations of the metric oscillates very slowly. Case (2) describes a scalar metric perturbations with increasing amplitude $2|C_1|t^{1/2}$ but stop oscillating for very large t. In this case the amplitude for the metric perturbations become very important with time in the IR sector. Thus, in the IR sector the amplitude for the metric fluctuations $\chi_{CG}=t^{-(p/2+1)}h_{CG}$ become

$$\langle\chi_{CG}^2\rangle\simeq\frac{t^{-(p+2)}}{6\pi^2}\int_0^{k_o(t)}dk\,k^2[\xi_k^{(1,2)}(t)]^2,\qquad(29)$$

which becomes

$$\langle\chi_{CG}^2\rangle\propto[\xi_k^{(1,2)}(t)]^2 t^{p-5}.\qquad(30)$$

Here $\xi_k^{(1,2)}(t)$ denotes the time dependent modes $\xi_k(t)$ for cases (1) and (2), respectively. Note that $\langle\chi_{CG}^2\rangle$ increases with time for $p>5$ in case (1), and for $p>4$ in case (2). Furthermore, the density fluctuations for the matter energy density is [7] $\delta\rho/\rho=-2\chi$, so that

$$\frac{\langle(\delta\rho)^2\rangle^{1/2}}{\rho}\Big|_{IR}\propto\langle\chi_{CG}^2\rangle^{1/2}.\qquad(31)$$

Since the metric and matter perturbations are anticorrelated $[\xi_k^{(1,2)}=-\phi_c(t)t^{-(p/2+1)}u_k^{(1,2)}]$, one can write the density fluctuations in terms of $u_k^{(1,2)}$ [see Eqs. (30) and (31)].

To summarize, a stochastic approach for the field that describes the gauge-invariant perturbations for the metric was developed. These fluctuations describes an effective Hamiltonian H_{eff} for an harmonic oscillator with an effective time-dependent parameter of mass k_o^2/a^2 and an external stochastic force ξ_c. Finally, in this report I demonstrated that the metric fluctuations can be very important on the IR sector, in a power-law expanding universe, when p is sufficiently large. Thus, in a power-law expansion with large p, one obtains large amplitude for scalar perturbations of the metric.

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